GENERALIZED GEOMETRIC MEANS AND ALGEBRAIC EQUATIONS

By Otto Dunkel

It has long been known that the arithmetic mean of n positive quantities is greater than or equal to their geometric mean and several proofs of this theorem have been given, one of which is due to Cauchy.* It has also been shown by Hamy that, if all the geometric means taking p of the n quantities at a time be added and divided by the number of means thus obtained, the result will be greater than or equal to the corresponding result taking $p + 1$ at a time; † thus showing that there is a descending series of mean values beginning with the arithmetic and ending with the geometric mean.

The object of this paper is to show that a theorem in regard to the roots of an algebraic equation ‡ leads almost immediately to the above theorem in regard to the relative magnitudes of the arithmetic and geometric means and at the same time establishes that there is another descending series of means intermediate in magnitude between these two and of a form different from those of Hamy.

The inequalities thus obtained may be used to derive sufficient conditions for imaginary roots of an algebraic equation, and a fairly simple set of such conditions will be worked out.

It will also be shown that all of these means can be represented as special values of a function of the n quantities and a continuous variable x, so that this function of x will be a continuous mean. A theorem in regard to this function will be used to derive the inequalities of Hamy and also a few other inequalities existing between the different kinds of means.

A Descending Series of Generalized Geometric Means. We shall prove the following theorem:

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* Cauchy, *Analyse algébrique* (1821), p. 457.

† M. Hamy, *Bull. des sciences math.*, ser. 2, vol. 14, part 1 (1890), p. 103. If we represent the number of combinations of n things taking p at a time by the symbol ${}_nC_p$, then the inequality above is:

$$\frac{\sum (m_1 m_2 m_3 \ldots m_p)^{\frac{1}{p}}}{{}_nC_p} \geqq \frac{\sum (m_1 m_2 m_3 \ldots m_{p+1})^{\frac{1}{p+1}}}{{}_nC_{p+1}} .$$

A proof of this is given on page 30 inequality (24) of this article.

‡ Dunkel, Annals of Mathematics, ser. 2, vol. 10 (1908), p. 48.

If $m_1, m_2, m_3, \cdots m_n$ are real and positive numbers, then:

$$(1)\qquad \frac{\Sigma m_1}{n} \geqq \left(\frac{\Sigma m_1 m_2}{{}_nC_2}\right)^{\frac{1}{2}} \geqq \left(\frac{\Sigma m_1 m_2 m_3}{{}_nC_3}\right)^{\frac{1}{3}} \geqq \cdots \geqq (m_1 m_2 m_3 \cdots m_n)^{\frac{1}{n}},$$

where $\Sigma m_1 m_2 m_3 \cdots m_i$ means the sum of all the products of the m's taking i of them at a time, and of which there are ${}_nC_i = \frac{n!}{i!(n-i)!}$; the equality sign holding for any two means not zero only when $m_1 = m_2 = m_3 = \cdots m_n$.

The n positive numbers $m_1, m_2, m_3, \cdots m_n$ are the roots of an algebraic equation of the nth degree, which it will be convenient to write with binomial coefficients as follows:

$$(2)\quad x^n - na_1x^{n-1} + {}_nC_2a_2x^{n-2} - \cdots + (-1)^i{}_nC_ia_ix^{n-i} + \cdots + (-1)^na_n = 0,$$

where ${}_nC_ia_i = \Sigma\, m_1m_2m_3 \cdots m_i$.

Let us suppose at first that no m is zero; then no member of the series (1) will be zero and each a_i in (2) will be positive and not zero. Since the roots of the equation (2) are all real, the relation

$$(3)\qquad a_i^2 \geqq a_{i-1}\, a_{i+1}, \qquad i = 1, 2, 3, \cdots n-1,$$

must be satisfied,* and this can be written as the continued inequality

$$(4)\qquad \frac{a_1}{1} \geqq \frac{a_2}{a_1} \geqq \frac{a_3}{a_2} \geqq \cdots \geqq \frac{a_i}{a_{i-1}} \geqq \frac{a_{i+1}}{a_i} \geqq \cdots \geqq \frac{a_{n-1}}{a_{n-2}} \geqq \frac{a_n}{a_{n-1}}.$$

The relations in (4) furnish the following inequalities:

$$(5)\qquad \begin{aligned} \frac{a_1}{1} &\geqq \frac{a_{i+1}}{a_i}, \\ \frac{a_2}{a_1} &\geqq \frac{a_{i+1}}{a_i}, \\ \frac{a_3}{a_2} &\geqq \frac{a_{i+1}}{a_i}, \\ &\;\;\vdots \\ \frac{a_i}{a_{i-1}} &\geqq \frac{a_{i+1}}{a_i}, \end{aligned}$$

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* Dunkel, loc. cit.

and, on taking the product of the left-hand sides and the right-hand sides, we have:

$$(6)\quad a_i \geqq \left(\frac{a_{i+1}}{a_i}\right)^i, \quad \text{or} \quad a_i^{i+1} \geqq a_{i+1}^i, \quad \text{or} \quad a_i^{\frac{1}{i}} \geqq a_{i+1}^{\frac{1}{i+1}}, \qquad i = 1, 2, 3, \cdots n-1.$$

Inserting the value of the a's given in (2) we have the first part of theorem (1) for the case in which no m is zero.

If some of the m's are zero, say $m_{s+1} = m_{s+2} = \cdots m_n = 0$, but no other m is zero; then in (1) all the expressions after the sth are zero, and in (2) the last a not zero is a_s, while in (4) the last ratio that we need consider is $\frac{a_s}{a_{s-1}}$. The reasoning then follows just as before.

If all of the m's are equal it is clear that the equality sign must be used in (1) throughout, and we have just seen that, if some of the m's are zero, the equality sign must be used for all the relations in (1) after a certain one. We shall now show that if two of the expressions in (1) are equal and not zero, then all the m's must be equal. To prove this it will suffice to show that, if not all the m's are equal, $a_1^2 > a_2$, since from this inequality it follows that in the first relation of (4) and of (5) the inequality sign alone must be used, if a_i is not zero, and finally the same thing must be true of (6).

That $a_1^2 > a_2$ can be seen to be true from the fact that, if all the roots of a polynomial are real and at least two are distinct, the same thing must be true of its first derivative; and repeating this reasoning on each derivative in turn we reach the conclusion that the $(n-2)$nd derivative which in this case is $x^2 - 2a_1 x + a_2$, if we drop off a numerical factor, must have real and unequal roots, and therefore the inequality stated above must be true.*

*This may also be shown directly as follows:

$$a_1^2 - a_2 = \frac{1}{n^2}\left[(\Sigma m_1)^2 - \frac{2n}{n-1}\Sigma m_1 m_2\right] = \frac{1}{n^2}\left[\Sigma m_1^2 - \frac{2}{n-1}\Sigma m_1 m_2\right] = \frac{1}{n^2(n-1)}\Sigma(m_1 - m_2)^2.$$

Therefore $a_1^2 - a_2$ is greater than zero, if at least two m's are not equal.

This completes the proof of theorem 1.*

Application to Algebraic Equations. It has been shown in a previous paper † that, if any equation written in the form:

$$(7)\qquad x^n + na_1x^{n-1} + {}_nC_2a_2x^{n-2} + \cdots + {}_nC_ia_ix^{n-i} + \cdots + na_{n-1}x + a_n = 0$$

has only real roots, then the following equation:

$$(8)\qquad a_{m-1}x^t + ta_mx^{t-1} + {}_tC_2a_{m+1}x^{t-2} + \cdots + {}_tC_ia_{m+i-1}x^{t-i} + \cdots + ta_{m+t-2}x + a_{m+t-1} = 0$$

has also only real roots, and therefore t real roots, if $a_{m-1} \neq 0$.

Let us suppose that all of the roots of (7) are real and also that $a_{m-1} \neq 0$, then, if we indicate the roots of (8) by $\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_t$, the numbers $\alpha_1^2, \alpha_2^2, \alpha_3^2, \cdots \alpha_t^2$ are all positive, and we can apply to them any one of the inequalities (1), which we have just proven. This can be very readily done since it is a very simple matter to obtain the equation whose roots are the squares of the roots of equation (8); ‡ and the coefficients of this

* In proving this theorem we made use of the fact that all of the m's and consequently all of the a's were positive; but (3) is true even if some of the m's are negative. Thus for any real quantities:

$$\left[\frac{\Sigma m_1 m_2 \ldots m_i}{{}_nC_i}\right]^2 - \left[\frac{\Sigma m_1 m_2 \ldots m_{i-1}}{{}_nC_{i-1}}\right]\left[\frac{\Sigma m_1 m_2 \ldots m_{i+1}}{{}_nC_{i+1}}\right] \geqq 0.$$

This can also be shown without the use of theorems on the roots of algebraic equations by a direct but somewhat tedious reduction of the left-hand side of this inequality to:

$$\frac{1}{(n-i)i^2}\left(\frac{1}{{}_nC_i}\right)^2 \Sigma \left\{ (m_1 - m_2)^2 \left[\Sigma(m_3m_4 \ldots m_{i-1})^2 + \frac{1}{{}_{i-1}C_1} \Sigma(m_3m_4 \ldots m_{i-2})^2(\Sigma m_{i-1})^2 + \ldots + \frac{1}{{}_{i-1}C_t} \Sigma(m_3m_4 \ldots m_t)^2(\Sigma m_{t+1}m_{t+2} \ldots m_{i-1})^2 + \ldots \right]\right\}.$$

This expression is always positive if the m's are real. The equality in the preceding footnote is the special case of this in which $i = 1$.

† Dunkel, loc. cit., p. 47. The above statement of the theorem is different from the original but is easily seen to be equivalent to it. From the proof given of the theorem it will be obvious that we might have used the reciprocal equation to (8), i. e., an equation with the coefficients of (8) in reverse order; and this fact will be used in connection with (11).

‡ Cf. Burnside and Panton, *Theory of Equations*, vol. 1, p. 78. If we represent the left-hand side of (8) by $f(x)$, then the equation whose roots are the squares of the roots of (8) may be written:

$$f(\sqrt{z}) \cdot f(-\sqrt{z}) = 0.$$

new equation furnish all the expressions such as $\Sigma \alpha_1^2 \alpha_2^2 \alpha_3^2 \cdots \alpha_t^2$ in terms of the a's of equation (7). In this way we should obtain a number of necessary conditions for the reality of all the roots of (7). We shall work out one of the simplest of these sets of conditions, making use of

$$(9) \qquad \frac{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_t^2}{t} \geqq (\alpha_1^2 \alpha_2^2 \cdots \alpha_t^2)^{\frac{1}{t}},$$

which follows from (1). From the equation of the squared roots of (8), or otherwise, we have:

$$\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_t^2 = \left(\frac{ta_m}{a_{m-1}}\right)^2 - 2\left(\frac{t(t-1)}{2}\frac{a_{m+1}}{a_{m-1}}\right)$$

$$= \frac{t^2}{a_{m-1}^2}\left(a_m^2 - \frac{t-1}{t} a_{m-1}a_{m+1}\right), \qquad \alpha_1^2\alpha_2^2\alpha_3^2 \cdots \alpha_t^2 = \left(\frac{a_{m+t-1}}{a_{m-1}}\right)^2,$$

and inserting these in (9) we have, after division by $\frac{t}{a_{m-1}^2}$:

$$(10) \qquad a_m^2 - \frac{t-1}{t} a_{m-1}a_{m+1} \geqq \frac{1}{t}\left[a_{m-1}^{1-\frac{1}{t}} a_{m+t-1}^{\frac{1}{t}}\right]^2, \qquad a_{m-1} \neq 0.$$

If we use instead of (8) the equation

$$(11) \quad a_{m+1}x^t + ta_m x^{t-1} + {}_tC_2 a_{m-1} x^{t-2} + \cdots + {}_tC_i a_{m-i+1} x^{t-i} + \cdots + a_{m-t+1} = 0,$$

which must also have t real roots, we shall find by the same reasoning:

$$(12) \qquad a_m^2 - \frac{t-1}{t} a_{m-1}a_{m+1} \geqq \frac{1}{t}\left[a_{m+1}^{1-\frac{1}{t}} a_{m-t+1}^{\frac{1}{t}}\right]^2, \qquad a_{m+1} \neq 0.$$

These results may be stated as follows: *If either of the inequalities*

$$(13) \qquad a_m^2 - \frac{t-1}{t} a_{m-1}a_{m+1} < \frac{1}{t}\left[a_{m-1}^{1-\frac{1}{t}} a_{m+t-1}^{\frac{1}{t}}\right]^2, \qquad a_{m-1} \neq 0,$$

$$< \frac{1}{t}\left[a_{m+1}^{1-\frac{1}{t}} a_{m-t+1}\right]^2, \qquad a_{m+1} \neq 0,$$

is satisfied for any values of m *and* t, *then the equation* (7) *has imaginary roots.*

This test may in special cases indicate the presence of imaginary roots when the test previously given*fails, that is, when $a_m^2 - a_{m-1}\, a_{m+1} \geqq 0$, and so may be regarded as a supplement to that test.

A Continuous Mean. Regarding $m_1, m_2, m_3, \cdots m_n$ as positive numbers no one of which is zero,

$$(14) \qquad y = \left(\frac{m_1^x + m_2^x + \cdots + m_n^x}{n}\right)^{\frac{1}{x}}$$

is a continuous function of the real variable x for all values of x except $x = 0$; and it will be seen later that y can be so defined for $x = 0$ as to be continuous without exception.†

It is clear that for any value of x for which it is defined, y is intermediate in value between the largest and smallest of the m's; also for $x = 1$ it is the arithmetic mean of the m's; for $x = -1$ it is the harmonic mean; and we shall now show that *y increases with x except when all the m's are equal; and that*

$$(15) \qquad \begin{aligned} &\lim_{x=-\infty} y = m_1, && \textit{where } m_1 \textit{ is as small as any other } m, \\ &\lim_{x=0} y = (m_1 m_2 m_3 \cdots m_n)^{\frac{1}{n}}, && \\ &\lim_{x=+\infty} y = m_n, && \textit{where } m_n \textit{ is as large as any other } m. \end{aligned}$$

The values of these limits can be found by putting them in the following form:

$$\operatorname{Lim} y = \operatorname{Lim} (e^{\log y});$$

so that it suffices to find the limit of $\log y$ in the several cases. Taking first the case of $x = 0$:

$$\lim_{x=0} \log y = \lim_{x=0} \frac{\log (m_1^x + m_2^x + \cdots + m_n^x) - \log (m_1^0 + m_2^0 + \cdots + m_n^0)}{x}$$

$$= \frac{d}{dx} \log(m_1^x + m_2^x + \cdots + m_n^x)\,\Big|_{x=0} = \log(m_1 m_2 m_3 \cdots m_n)^{n}.$$

Therefore, $$\lim_{x=0} y = (m_1 m_2 m_3 \cdots m_n)^{\frac{1}{n}}.$$

* Dunkel, loc. cit., p. 48.

† Only real and positive values of m_i^x and of y are considered.

In order to find the limit for $x = -\infty$, we have supposed that m_1 is as small as any other m; then:

$$\operatorname*{Lim}_{x=-\infty} \log y = \log m_1 + \operatorname*{Lim}_{x=-\infty}\left(\frac{1}{x}\log\frac{1+\left(\frac{m_2}{m_1}\right)^x+\cdots+\left(\frac{m_n}{m_1}\right)^x}{n}\right)$$

$$= \log m_1, \qquad\qquad \text{where} \quad \frac{m_i}{m_1} \geqq 1,$$

since the limit on the right in the first line is zero. This completes the proof of the first two limits in (15); and the third limit is found by replacing x by $-x$ and proceeding as before.

It remains now to show that y increases with x; this will be done by showing that

$$(16) \qquad \frac{dy}{dx} = \frac{y}{x}\left[\frac{\Sigma m_i^x \log m_i}{\Sigma m_i^x} - \frac{1}{x}\log\left(\frac{\Sigma m_i^x}{n}\right)\right], \quad x \neq 0,$$

is always positive. Let us suppose then that the value of x is fixed; to simplify the proof we shall write

$$m_i^x = b_i, \qquad \text{or} \qquad m_i = b_i^{\frac{1}{x}},$$

and then (16) becomes:

$$(16') \qquad \frac{dy}{dx} = \frac{y}{x^2}\,\frac{1}{\Sigma b_i}\left[\Sigma b_i \log b_i - \Sigma b_i \cdot \log\left(\frac{\Sigma b_i}{n}\right)\right].$$

Since the part outside of the square brackets is always positive, it is only necessary to show that the expression within is positive. Remembering that $b_1, b_2, b_3, \cdots b_r$ are fixed, we shall examine the following function of t, where t is positive:

$$(17) \qquad u = \sum_1^r b_i \log b_i + t\log t - \left(\sum_1^r b_i + t\right)\log\left(\frac{\sum_1^r b_i + t}{r+1}\right)$$

for a minimum. Its derivative with respect to t is

$$(18) \qquad \frac{du}{dt} = \log t - \log\left(\frac{\sum_1^r b_i + t}{r+1}\right) = \log\left(\frac{r+1}{\frac{1}{t}\sum_1^r b_i + 1}\right),$$

and therefore

$$\text{(18}')\qquad \frac{du}{dt} \lesseqgtr 0 \quad \text{according as} \quad \frac{1}{t}\sum_1^r b_i \gtreqless r \quad \text{or as} \quad t \lesseqgtr \frac{1}{r}\sum_1^r b_i;$$

and this tells us that u has for $t = \frac{1}{r}\sum_1^r b_i$ its minimum value, which after reduction is

$$\text{(19)}\qquad u_r = \sum_1^r b_i \log b_i - \sum_1^r b_i \log\left(\frac{1}{r}\sum_1^r b_i\right).$$

We have then, putting $t = b_{r+1}$ in (17):

$$\text{(20)}\qquad u_{r+1} \geqq u_r \quad \text{according as} \quad b_{r+1} \underset{=}{\neq} \frac{1}{r}\sum_1^r b_i\,.$$

Giving r in turn the values 1, 2, 3, $\cdots$ n, and noting that $u_1 = 0$, we have:

$$\text{(20}')\qquad u_n \geqq u_{n-1} \geqq u_{n-2} \geqq \ldots \geqq u_3 \geqq u_2 \geqq u_1 = 0.$$

If there are two b's not equal, we may consider them as b_1 and b_2; and from (20) we see that in this case $u_2 > u_1 = 0$, and therefore u_n is greater than zero. Since u_n is the expression in the brackets of (16′), this completes the proof that, if not all the b's are equal, i. e., if not all the m's are equal, $\frac{dy}{dx}$ is always positive. This also furnishes the proof of (15).

If then for $x = 0$ we assign to y the value of the geometric mean of all the m's, it is easily seen that this completed definition makes *y a continuous and increasing function for all values of x.*

The case in which some of the m's are zero can now be easily treated; for, if we suppose that there are $n + k$ of the m's of which the first n are each different from zero but $m_{n+1} = m_{n+2} = \cdots = m_{n+k} = 0$, then (14) may be written:

$$\text{(21)}\qquad y = \left(\frac{m_1^x + m_2^x + \cdots + m_n^x}{n}\right)^{\frac{1}{x}} \left(\frac{n}{n+k}\right)^{\frac{1}{x}}, \quad x > 0,$$

and we may define y as zero when $x \leqq 0$.

The first factor in the first expression for y has already been shown to increase with x, and it is easily seen that the second factor also increases with

x and approaches unity as x becomes infinite through positive values, but approaches zero as x approaches zero through positive values. Hence in this case y increases with x for positive values of x and approaches the largest m as x increases indefinitely.

From what preceeds it will be seen that if we define a function of x as follows:

$$(22) \qquad y = \left(\frac{m_1^x + m_2^x + \cdots + m_n^x}{p}\right)^{\frac{1}{x}},$$

where none of the m's are zero and $p \neq n$, this function of x will be discontinuous for $x = 0$ owing to the discontinuity of $\left(\frac{n}{p}\right)^{\frac{1}{x}}$ at this point.

Derivation of Inequalities between Several Forms of Means. In what follows we shall suppose that none of the m's in the expression (14) for y are zero; and in this case, if x and y were plotted, we would obtain a continuous curve rising constantly from the least m at $-\infty$ to the greatest m at $+\infty$ and passing through the harmonic mean at -1, the geometric mean at 0, and the arithmetic mean at $+1$. Also y must take on each and every value between the least and greatest m once and only once. It follows then that each of the means which have been mentioned here, and indeed any mean whatsoever of these n numbers, must be represented by a single point on this curve. After proving the inequalities of Hamy, * we shall derive certain other inequalities which give some idea of the order in which the different means are located on the curve.

In order to prove that:

$$(23) \qquad \frac{\Sigma m_1}{n} \geqq \frac{\Sigma (m_1 m_2)^{\frac{1}{2}}}{{}_nC_2} \geqq \frac{\Sigma (m_1 m_2 m_3)^{\frac{1}{3}}}{{}_nC_3} \geqq \cdots \geqq (m_1 m_2 m_3 \cdots m_n)^{\frac{1}{n}}$$

we shall apply (15) which tells us that

$$\left[\frac{\Sigma (m_1 m_2 m_3 \cdots m_p)^{\frac{1}{p}}}{{}_nC_p}\right]^p \geqq \left[\frac{\Sigma (m_1 m_2 m_3 \cdots m_p)^{\frac{1}{p+1}}}{{}_nC_p}\right]^{p+1};$$

but from (1) we have, after replacing m_i by $m_i^{\frac{1}{p+1}}$,

$$\left[\frac{\Sigma (m_1 m_2 m_3 \cdots m_p)^{\frac{1}{p+1}}}{{}_nC_p}\right]^{\frac{1}{p}} \geqq \left[\frac{\Sigma (m_1 m_2 m_3 \cdots m_{p+1})^{\frac{1}{p+1}}}{{}_nC_{p+1}}\right]^{\frac{1}{p+1}}.$$

* Loc. cit.

From these two inequalities we have

$$(24)\qquad \frac{\Sigma(m_1m_2m_3\cdots m_p)^{\frac{1}{p}}}{{}_nC_p} \geqq \left[\frac{\Sigma(m_1m_2m_3\cdots m_p)^{\frac{1}{p+1}}}{{}_nC_p}\right]^{\frac{p+1}{p}}$$
$$\geqq \frac{\Sigma(m_1m_2m_3\cdots m_{p+1})^{\frac{1}{p+1}}}{{}_nC_{p+1}},$$

and this gives the relations in (23).

From (1) it follows on replacing m_i by $m_i^{\frac{1}{p}}$ that:

$$(25)\qquad \left[\frac{m_1^{\frac{1}{p}}+m_2^{\frac{1}{p}}+m_3^{\frac{1}{p}}+\cdots+m_n^{\frac{1}{p}}}{n}\right]^p \geqq \frac{(\Sigma m_1m_2m_3\cdots m_p)^{\frac{1}{p}}}{{}_nC_p}.$$

Corresponding members of (1) and (23) may be compared as follows:

$$(26)\qquad \left[\frac{\Sigma m_1m_2m_3\cdots m_p}{{}_nC_p}\right]^{\frac{1}{p}} = \left[\frac{\Sigma\left[(m_1m_2m_3\cdots m_p)^{\frac{1}{p}}\right]^p}{{}_nC_p}\right]^{\frac{1}{p}}$$
$$\geqq \frac{\Sigma(m_1m_2m_3\cdots m_p)^{\frac{1}{p}}}{{}_nC_p},$$

the inequality in the above following from (15), since $p \geqq 1$.

Thus each member of (1) is greater than or equal to the corresponding member of (23).

It would be interesting to have other inequalities corresponding to (25) which would show in what intervals on the x axis each member of (1) and (23) would appear when plotted on the curve of (14).

Approximations to the Roots of Algebraic Equations. If it is known that all the roots of an algebraic equation are real, the properties of the function y in (14) may be used to obtain approximations to their absolute values. Let us suppose that the roots arranged in order of their numerical magnitude are m_1, m_2, m_3, $\cdots$ m_n, so that m_n is as large in absolute value as any other root. If now x takes on only even integral values, the results in (15) will still be true with a slight modification, whether all of the roots are positive or not, and this modification consists in replacing the m's by their absolute values in the values of the different limits; thus two of the limits in (15) would now read:

$$\lim_{x=-\infty} y = |m_1|\,; \qquad \lim_{x=+\infty} y = |m_n|\,.$$

If then to x is assigned the series of values 2, 4, 6, 8, $\cdots$, there will result a set of increasing values:

$$y_2 < y_4 < y_8 < \cdots,$$

which will approach $|m_n|$ as a limit. On the other hand, corresponding to the values -2, -4, -6, -8, $\cdots$ of x, there is a descending series:

$$y_{-2} > y_{-4} > y_{-6} > y_{-8} > \cdots$$

approaching $|m_1|$ as a limit.

Considering next the following function of x:

$$\left[\frac{\Sigma(m_1 m_2)^x}{{}_nC_2}\right]^{\frac{1}{x}}, \tag{27}$$

we would obtain in the same way a set of increasing values corresponding to increasing values of x and approaching $|m_{n-1}|\,|m_n|$ as a limit. Taking the quotient of (14) by (27) we obtain a function of x:

$$\left[\frac{\Sigma(m_1 m_2)^x}{\Sigma m_1^x}\,\frac{n}{{}_nC_2}\right]^{\frac{1}{x}},$$

which will approach $|m_{n-1}|$ as a limit; and in general

$$\left[\frac{\Sigma(m_1 m_2 \,.\,.\,.\, m_p)^x}{\Sigma(m_1 m_2 \,.\,.\,.\, m_{p-1})^x}\,\frac{p}{n-p+1}\right]^{\frac{1}{x}} \tag{28}$$

will approach $|m_{n-p+1}|$ as a limit as x increases indefinitely.

The numerical values of the expressions $\Sigma(m_1\, m_2 \cdots m_p)^x$ in the successive approximations (28) can be obtained most easily, if we consider only those values of x which are powers of 2, by forming the equation whose roots are the squares of the roots of the original equation; * then the equation whose roots are the squares of the roots of the equation just found and so on: so that if the tth equation thus found is:

$$z^n + C_1^{(t)} z^{n-1} + C_2^{(t)} z^{n-2} + \,.\,.\,.\, + C_{n-1}^{(t)} z + C_n^{(t)} = 0,$$

* Cf. footnote ‡ page 24.

then its roots are $m_1^{2^t}, m_2^{2^t}, m_3^{2^t} \cdots$ and the numerical value of $\Sigma(m_1 m_2 m_3 \cdots m_p)^{2^t}$ is $(-1)^p C_p^{(t)}$. Inserting these values in (28) we have:

$$\left[-\frac{C_p^{(t)}}{C_{p-1}^{(t)}} \frac{p}{n-p+1}\right]^{\frac{1}{2^t}} \tag{29}$$

as an approximation to $|m_{n-p+1}|$. Forming all the n expressions (29) corresponding to the given equation, all of its roots may in this way be approximated simultaneously.*

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* Cf. Netto, *Algebra*, vol. 1, p. 290–297. On page 292 the approximations to the three roots of a cubic are worked out. The expressions in Netto corresponding to (29) above do not contain the factor $\frac{p}{n-p+1}$.